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2010 J. Phys. A: Math. Theor. 43 115207

(http://iopscience.iop.org/1751-8121/43/11/115207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 115207 (9pp) doi:10.1088/1751-8113/43/11/115207

Kapteyn series in high intensity Compton scattering

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Received 9 November 2009, in final form 14 January 2010
Published 2 March 2010
Online at stacks.iop.org/JPhysA/43/115207

Abstract
In discussing the signatures available from high intensity Compton scattering, a problem of considerable interest in photon scattering from active galactic nuclei and their emission jets, Harvey *et al* (2009 *Phys. Rev.* A **79** 063407) showed that the properties of some other than conventional Kapteyn series play fundamental roles in determining the spectral output both with respect to frequency and with respect to emission angle. While they were able to provide bounds to the required series in terms of known Kapteyn series that could be summed analytically in closed form, only numerical analysis could take their investigation further as they demonstrated. The purpose of this paper is to show that the many Kapteyn series involved in the scattering problem can all be reduced either to analytic form or to a single Kapteyn series that cannot be evaluated in closed form but for which an integral representation is available. This reduction is of considerable benefit in controlling the correctness and accuracy of numerical investigations; the reduction also provides significant insight into some basic procedures for summing such unconventional Kapteyn series as well as allowing a better understanding of the dependence of the scattering on the physical parameters involved than would be the case directly from the series.

PACS number: 34.50.−s

1. Introduction

The evaluation of Compton scattering from high intensity radiation is a subject that has recently come under intense scrutiny (Harvey *et al* 2009) for many reasons not the least of which is the application of such scattering to active galactic nuclei (AGNs) and the corresponding emission jets (see, e.g., Krolik (1998) and Rees (1984) for an introduction to the subject). The understanding of Compton scattering is also important for the production of high-energy cosmic ray particles in AGNs (Biermann *et al* 2009) and, with relation to observational issues, for the identification and for distinguishing different AGN spectral properties, as for example shown by Tozzi *et al* (2006) for the Chandra Deep Field South. A related application of Compton scattering is the heating and cooling of electrons (the so-called radiative feedback mechanism), which is important if one wants to understand the effect AGNs have on galaxy formation and evolution (Yuan *et al* 2009). Furthermore, the interpretation of variability studies and flare emission in Sgr A* depends on inverse Compton scattering that has to be understood first (Yusef-Zadeh *et al* 2006).

Although the application of Harvey *et al* (2009) that motivated this article was more in the realm of laboratory high intensity Compton scattering, their procedure is, nevertheless, precisely that which is required in the astrophysical domain. In evaluating the processes involved, one is confronted with a slew of summations of Kapteyn series (Kapteyn 1893) of the second kind that involve the product of two Bessel functions in which the arguments are of the form nz, with z being a fixed parameter and n the summation index (see Watson (1966)). In addition, there are coefficients multiplying the Bessel functions that are also dependent on the summation index n.

It is of considerable interest to attempt to sum such series so that the structural dependence of the desired radiation spectra can be provided in a more cogent manner than is available from the infinite Kapteyn series representations. While Harvey *et al* (2009) were able to provide limits to the relevant series in terms of Kapteyn series that are analytically available in closed form, they also noted that they had been unable to perform the required series in closed form—a necessary step on the road to showing how the radiation spectra are influenced by the various parameters entering the high intensity scattering problem. The alternative is a brute force numerical evaluation which suffers from the drawbacks that one does not have a clear picture of general dependences nor is it that simple to determine the numerical accuracy of such results—a point emphasized in considerable detail by Harvey *et al* (2009).

The purpose of this paper is to show that the many Kapteyn series of the second kind involved in the scattering problem can all be reduced to the evaluation of just a single Kapteyn series but that this last Kapteyn series cannot be reduced to analytic form rather just to an integral representation. However, such information is also of great benefit for one can use the integral representation and the series representation to control the numerical accuracy. In addition one can see better the parameter dependence of the integral and one can also make asymptotic expansions of the integral under controlled conditions, something which is not as easy to do or justify when one has infinite series involved. One can also determine the range of convergence of integrals somewhat better than Kapteyn series and one can also estimate residual corrections in a cleaner way. For all these reasons it is appropriate to discuss the methods and procedures required to bring the many Kapteyn series involved in the scattering problem to some form of order. It may also be that such a discussion will be of use in other problems involving Kapteyn series, which is another powerful incentive to give such an investigation here.

2. Technical development

2.1. Series connections

There are two fundamental groups of Kapteyn series involved in the high intensity scattering problem. These groups are

$$S_{N,M}(x,z)=\sum_{n=1}^{\infty}\frac{n^N}{(1+nx)^M}J_n^2(nz) \tag{1}$$

and

$$T_{N,M}(x,z)=\sum_{n=1}^{\infty}\frac{n^N}{(1+nx)^M}J_n'^2(nz),\tag{2}$$

with $N\in\{1,2,3\}$ and, independently, $M\in\{2,3\}$. Furthermore, J_n' is the derivative of the Bessel function J_n with respect to the argument, nz.

There is a close connection between the groups $S_{N,M}$ and $T_{N,M}$ as follows. Consider

$$F(z)=\sum_{n=1}^{\infty}a(n)J_n^2(nz)\tag{3}$$

and

$$G(z)=\sum_{n=1}^{\infty}a(n)J_n'^2(nz),\tag{4}$$

where $a(n)$ is arbitrary but known.

Then

$$\frac{\mathrm{d}G}{\mathrm{d}z}=2\sum_{n=1}^{\infty}na(n)J'n(nz)J_n''(nz).\tag{5}$$

Use Bessel's equation in the form

$$J_n''(nz)=\frac{1}{z^2}\left[(1-z^2)J_n(nz)-\frac{z}{n}J_n'(nz)\right]\tag{6}$$

when

$$\frac{\mathrm{d}G}{\mathrm{d}z}=-\frac{2}{z}G+\frac{1-z^2}{z^2}\frac{\mathrm{d}F}{\mathrm{d}z}.\tag{7}$$

Hence if F is known in closed form, then one has

$$G(z)=\frac{1}{z^2}\left[(1-z^2)F+2\int_0^z\mathrm{d}u\;uF(u)\right].\tag{8}$$

Conversely, if G is known in closed form, then

$$F(z)=\frac{z^2}{1-z^2}G-2\int_0^z\mathrm{d}u\;\frac{u^3}{(1-u^2)^2}G(u).\tag{9}$$

Thus, it is only necessary to evaluate in closed form either $S_{N,M}$ or $T_{N,M}$ with $T_{N,M}$ or $S_{N,M}$ being given as a quadrature, respectively.

2.2. *Reduction of the series $S_{N,M}$*

Consider

$$Q_N(y,z)=\sum_{n=1}^{\infty}\frac{n^N}{n+y}J_n^2(nz),\tag{10}$$

with $y=1/x$. It is sufficient to consider Q_N because

$$\frac{\partial^{M-1}Q_N}{\partial y^{M-1}}=(-1)^{M-1}(M-1)!\sum_{n=1}^{\infty}\frac{n^N}{(n+y)^M}J_n^2(nz)\tag{11a}$$

$$\equiv(-1)^{M-1}(M-1)!\;x^M\,S_{N,M}(x,z),\tag{11b}$$

with again $x=1/y$. Now for Q_N there are three values of N to consider: $N=1$, 2 and 3.

Then with $y = 1/x$ one has

$$Q_1(y,z) = \sum_{n=1}^{\infty} J_n^2(nz) - y \sum_{n=1}^{\infty} \frac{J_n^2(nz)}{n+y} \tag{12a}$$

$$Q_2(y,z) = \sum_{n=1}^{\infty} n J_n^2(nz) - y \sum_{n=1}^{\infty} J_n^2(nz) + y^2 \sum_{n=1}^{\infty} \frac{J_n^2(nz)}{n+y} \tag{12b}$$

$$Q_3(y,z) = \sum_{n=1}^{\infty} n^2 J_n^2(nz) + y^2 \sum_{n=1}^{\infty} J_n^2(nz) - y \sum_{n=1}^{\infty} n J_n^2(nz) - y^3 \sum_{n=1}^{\infty} \frac{J_n^2(nz)}{n+y}. \tag{12c}$$

Several of the components of Q_1, Q_2 and Q_3 are known in closed form. Schott (1912) showed that

$$\sum_{n=1}^{\infty} J_n^2(nz) = \frac{1}{2}\left(\frac{1}{\sqrt{1-z^2}} - 1\right). \tag{13}$$

Equally, one has

$$\sum_{n=1}^{\infty} n^2 J_n^2(nz) = \frac{z^2(4+z^2)}{16(1-z^2)^{7/2}} \tag{14}$$

leaving, therefore, the two basic series

$$A(z) = \sum_{n=1}^{\infty} n J_n^2(nz) \tag{15}$$

and

$$B(x,z) = \sum_{n=1}^{\infty} \frac{J_n^2(nz)}{1+nx}, \tag{16}$$

to be evaluated. Now

$$\left.\frac{\partial B}{\partial x}\right|_{x=0} = -\sum_{n=1}^{\infty} n J_n^2(nz) = -A, \tag{17}$$

so that it is both necessary and sufficient to evaluate the series B.

2.3. The series B

Write the series B in the form

$$B(z) = \int_0^{\infty} \mathrm{d}u\, \mathrm{e}^{-u} \sum_{n=1}^{\infty} \mathrm{e}^{-unx} J_n^2(nz). \tag{18}$$

Now use the well-known representation (e.g. Gradshteyn and Ryzhik 2000)

$$J_n^2(nz) = \frac{2}{\pi^2}(-1)^n \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\psi [\cos(2na_+) + \cos(2na_-)] \tag{19}$$

with

$$a_\pm = z\cos\theta \sin\psi \pm \theta \tag{20}$$

when one can write

$$B(x,z) = \frac{2}{\pi^2}\int_0^\infty \mathrm{d}u\ \mathrm{e}^{-u}\int_0^{\pi/2}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\psi \\ \times\sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{-unx}\,[\cos(2na_+)+\cos(2na_-)]\,. \tag{21}$$

Because

$$\sum_{n=1}^{\infty}(-1)^n\mathrm{e}^{-unx}\cos(2na_\pm) = -\frac{\mathrm{e}^{-ux}[\cos(2a_\pm)+\mathrm{e}^{-ux}]}{1+2\mathrm{e}^{-ux}\cos(2a_\pm)+\mathrm{e}^{-2ux}} \tag{22a}$$

$$\equiv \frac{1}{2x}\frac{\partial}{\partial u}\ln[1+2\mathrm{e}^{-ux}\cos(2a_\pm)+\mathrm{e}^{-2ux}], \tag{22b}$$

then

$$B(x,z) = \frac{1}{\pi^2 x}\int_0^{\pi/2}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\psi\ (J_- + J_+) \tag{23}$$

with

$$J_\pm = \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u}\frac{\partial}{\partial u}\ln[1+2\mathrm{e}^{-ux}\cos(2a_\pm)+\mathrm{e}^{-2ux}]. \tag{24}$$

With $\mathrm{e}^{-u} = q$ one has

$$J_\pm = \int_0^1 \mathrm{d}q\ \ln[1+2\cos(2a_\pm)q^x+q^{2x}] - \ln[2+2\cos(2a_\pm)], \tag{25}$$

thus simplifying the integration of equation (23).

Note that the series given in equation (22*a*) is conditionally convergent in the sense that for all positive x (excluding $x = 0$) the given summation is correct. However, on precisely $x = 0$ the series consists of delta functions so that one is reduced to using the Schott formula for the series given in equation (16) for that unique value of $x = 0$. For all other values of positive x the closed form expression (22*a*) is valid. And the derivative $\mathrm{d}B/\mathrm{d}x$ evaluated on $x = 0$ is also valid as obtained from equation (22*a*) and provides an elliptic integral representation for the series A given through equation (15), which integral representation is precisely that given in Tautz and Lerche (2009).

While the integrals are not expressible in closed form for arbitrary values of x, they have several advantages over the direct representation of the various Kapteyn series. First, accurate numerical evaluation of integrals is considerably simpler to control than infinite series evaluation. Second, one can see immediately the structural dependence on the relevant parameters, something that is difficult to do with infinite series. Third, asymptotic evaluation of integrals for small or large parameter values is a mature subject so one can quickly determine relevant behavior of the many Kapteyn series involved in the high intensity Compton scattering.

2.4. Numerical comparison

In order to demonstrate that the integral representation of the Kapteyn series B is the same as the direct series, it is appropriate to consider a few numerical examples. First, note that from the physical perspective of the high intensity Compton scattering viewpoint it makes little sense to evaluate the series B for any value of $z > 1$. This limitation arises because the various series that are analytically available in equations (12*a*), (12*b*) and (12*c*) have limits of $z < 1$ for convergence. Accordingly, in order to illustrate the behavior of the series B in respect of the integral representation it suffices to present one value of z. We have chosen $z = 0.1$ although other values of z have also been investigated with equally good agreement for the series and integral representation. Second, note that there is no limitation on the value of $x > 0$ because the factor $1+nx$ in the denominator of the series for B enhances convergence of the series over the situation with $x = 0$. Two simple cases are presented: (1) when z is held fixed at $z = 0.1$ and x is allowed to vary (see figure 1); (2) when x is held fixed (at $x = 0.1$) and z is allowed to vary (see figure 2). Numerical evaluation of the series B was accomplished in two ways: by direct summation of the series and by direct evaluation of the integral representation. The numerical evaluation of the infinite sum is carried out as follows: first, a number of terms (usually 100) are summed directly; to accelerate the convergence of the sum, then Wynn's epsilon method (see, e.g., Brezinsky 2000, Hamming 1986) is used, which samples a number of additional terms (usually 100) in the sum, and then fits them to a polynomial multiplied by a decaying exponential. Thus, the series are well approximated and the required computer time is kept moderate. The convergence of the sums, in addition, is guaranteed by analytical considerations. Furthermore, numerical integrations are carried out using standard techniques such as adaptive step sizes (e.g. Press *et al* 2007).

However, some care has to be taken with the logarithmic singularity in the integral representation. Because Mathematica® version 7.0 is used, this problem is dealt with automatically. Using other packages, however, appropriate measures would have to be taken manually.

The degree to which the direct series evaluation and the integral representation are in agreement is measured by ΔB defined (in percent) as

$$\Delta B \equiv 100 \left| \frac{\text{Summation} - \text{Integral}}{\text{Summation}} \right| \tag{26}$$

and is evaluated for each value of z and x used in the numerical work. For each of the two illustrative cases (see figures 1 and 2), the deviation is always less than 0.05%; of course, such accuracy depends on the exact numerical details. The examples illustrate nonetheless the excellent agreement between the direct summation of the Kapteyn series B from equation (16) and the integral representation of equation (23).

In the astrophysical application of Harvey *et al* (2009, their section IV.C), namely the calculation of photon emission rates as a function of the scattering angle, expressions appear that are composed of the sums $S_{N,M}(x, z)$ and $T_{N,M}(x, z)$ and not just of the sum B. A referee noted that, to get there, additional steps—mostly taking derivatives of the series $Q_N(y, z)$, see section 2.2—have to be executed before the calculation of emission rates can be done, which requires a much higher level of convergency. However, any derivative with respect to $y = 1/x$ can be carried out inside the integral of the functions $J_\pm$, equation (24), *before* performing the integration. Because taking derivatives can always be done analytically, one is always left with the three numerical integrals in equation (23). Although such may be time-consuming, the benefit is that one has to worry only about the convergence of the integrals instead about that of the Kapteyn series, the latter subject being less understood and less mature than numerical integration. Furthermore, our intention is to both enable and encourage the interested reader in evaluating Kapteyn series that are more complicated than the basic types that have been known for almost a century now. Recent successful calculations of various Kapteyn series show that such is indeed possible. To serve as an example, the series $S_{1,2}$, which appears in Harvey *et al* (2009, their equation (78)), will be evaluated. Using equation (11*b*), one obtains

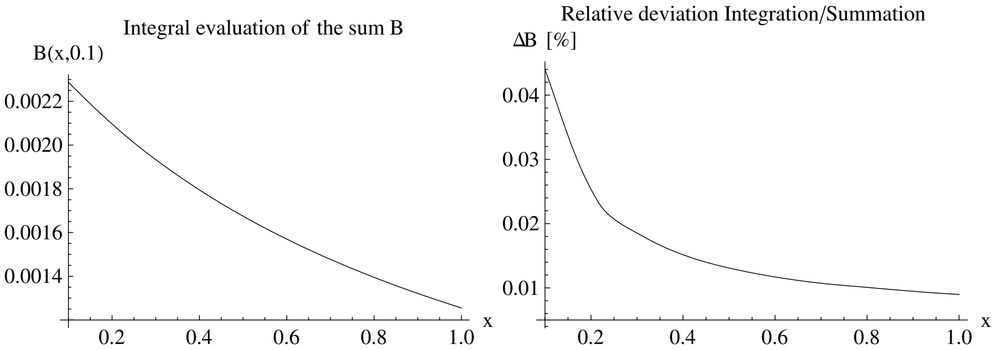

Figure 1. Left panel: the series B from equation (16) for varying x with $z = 0.1$ being held fixed. Right panel: the relative deviation ΔB between direct summation and evaluation of the integral representation from equation (23), as defined in equation (26).

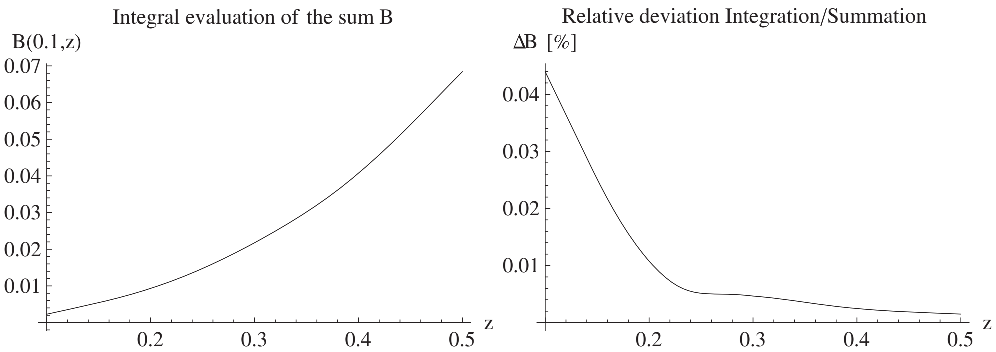

Figure 2. Left panel: the series B from equation (16) for varying z with $x = 0.1$ being held fixed. Right panel: the relative deviation ΔB between direct summation and evaluation of the integral representation from equation (23), as defined in equation (26).

$$S_{1,2}(x, z) = -\frac{\partial B}{\partial x} = \frac{1}{x}\, B(x, z) - \frac{1}{\pi^2 x} \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\psi\, \frac{\partial}{\partial x}\, (J_- + J_+) \tag{27}$$

with

$$\frac{\partial J_\pm}{\partial x} = \frac{2q^x \ln q\, [\cos(2a_\pm)]}{1 + 2\cos(2a_\pm) q^x + q^{2x}}. \tag{28}$$

The results are shown in figures 3 and 4 for varying x and z, respectively, showing again excellent agreement. Note that, for the numerical evaluation of the series $S_{1,2}$, the same accuracy goals as for the series B have been used.

3. Summary and discussion

While the problem of high intensity Compton scattering in laboratory experiments provided the initial driving force to reduce the Kapteyn series involved to some form of order so that analytical techniques could be brought to bear to aid in the understanding of the spectral

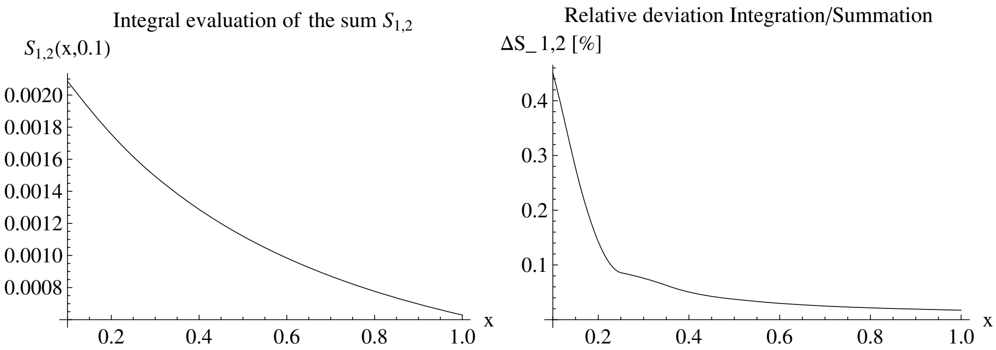

Figure 3. Left panel: the series $S_{1,2}$ from equation (27) for varying x with $z = 0.1$ being held fixed. Right panel: the relative deviation ΔB between direct summation and evaluation of the integral representation from equation (23), as defined in equation (26).

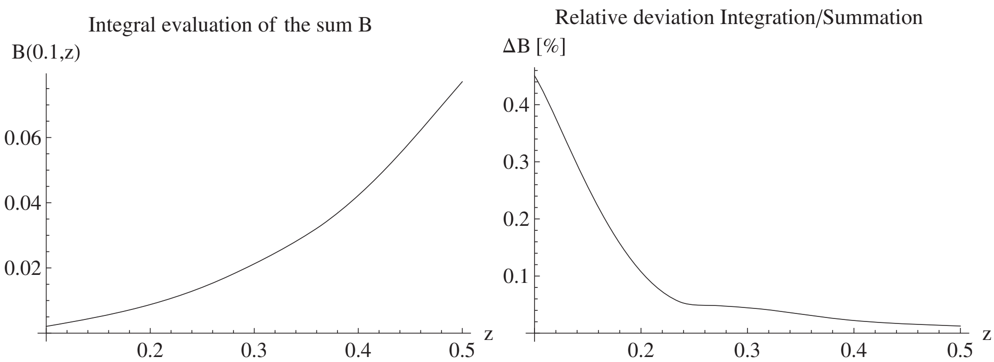

Figure 4. Left panel: the series $S_{1,2}$ from equation (27) for varying z with $x = 0.1$ being held fixed. Right panel: the relative deviation ΔB between direct summation and evaluation of the integral representation from equation (23), as defined in equation (26).

results, the motivation for this paper is also underscored by astrophysical problems of high intensity output from astrophysical objects.

The combination of such problems requires that one be able to address the mathematical structure of the groups of Kapteyn series involved in a way that allows one to evaluate better both the functional dependence of such series on the parameters involved and also the general dominance of individual parameters in controlling the overall patterns of behavior.

For the high intensity scattering problem a little effort shows that all except one of the many Kapteyn series involved can be reduced to closed analytic form, thereby facilitating the determination of the spectral behavior. The last remaining Kapteyn series involved is not analytically tractable but even then can be reduced to an integral representation so that it, too, allows one to better investigate the dependence on parameters involved. A couple of numerical examples indicated the close agreement between the integral representation and the Kapteyn series (to better than about a fraction of 10^{-4}) and other cases (not displayed) show similar accuracy.

The point has now been reached where, depending on each individual application, one can immediately use the Kapteyn series procedures and results presented here without the need for massive computer numerical work that would, in any event, mask the structural properties and interplay of the various Kapteyn series involved in high intensity Compton scattering, and that has been one of the main motivational aspects that helped formulate much of the work presented here.

Acknowledgments

This work was partially supported the Deutsche Forschungsgemeinschaft (DFG) through grant no Schl 201/19-1 and by the German Academy of Natural Scientists Leopoldina Fellowship Programme through grant LDPS 2009-14 funded by the Federal Ministry of Education and Research (BMBF). The authors also wish to thank C Harvey, T Heinzl and A Ilderton who were not only helpful in improving the paper but also noted a significant error in a prior version.

References

Biermann P *et al* 2009 *Nucl. Phys.* B **190** 61
Brezinski C 2000 *J. Comput. Appl. Math.* **122** 1
Gradshteyn I S and Ryzhik I N 2000 *Table of Integrals, Series, and Products* (London: Academic)
Hamming R W 1986 *Numerical Methods for Scientists and Engineers* 2nd edn (New York: Dover) pp 206–7
Harvey C, Heinzl T and Ilderton A 2009 *Phys. Rev.* A **79** 063407
Kapteyn W 1893 *Annales scientifiques de l'École Normale Supérieure Sér. 3* **10** 91
Krolik J H 1998 *Active Galactic Nuclei* (Princeton: Princeton University Press) section 9.2.5
Press W H, Teukolsky S A, Vetterling W T and Flannery B P 2007 *Numerical Recipes* (Cambridge: Cambridge University Press) pp 196–200
Rees M J 1984 *Ann. Rev. Astron. Astrophys.* **22** 471
Schott G A 1912 *Electromagnetic Radiation* (Cambridge: Cambridge University Press) chapter 8, p 120
Tautz R C and Lerche I 2009 *Adv. Math. Phys.* submitted (http://www.hindawi.com/journals/amp/aip.425164.html)
Tozzi P, Gilli R and Mainieri V *et al* 2006 *Astron. Astrophys.* **451** 457
Watson G N 1966 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press)
Yuan F, Xie F and Ostriker J P 2009 *Astrophys. J.* **691** 98
Yusef-Zadeh F, Bushouse H and Dowell C D *et al* 2006 *Astrophys. J.* **644** 198